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A SELF INVERSIVE ARRAY PROCESSING SCHEME FOR ANGLE-OF-ARRIVAL ESTIMATION*

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Abstract. This paper proposes an improved high-resolution technique that incorporates the forward (original) array together with its complex conjugated backward version to achieve superior performance. This is in turn realized by averaging the forward covariance matrix and the backward one, and using it in conjunction with the MUSIC technique. This paper also presents the asymptotic analysis in terms of first-order approximations of the mean and variance of the null spectrum estimator. It is shown here that in an uncorrelated scene, the bias corresponding to this scheme is exactly half of that associated with the conventional one while maintaining the same variance. Furthermore, the bias expression in the new scheme is used to obtain a resolution threshold for two uncorrelated, equipowered plane waves in white noise and the result is compared with that obtained by the conventional scheme for the same source scene.

Zusammenfassung. Dieser Beitrag schlägt ein verbessertes, hochauflösendes Spektralanalyseverfahren vor, bei dem das Vorwärts-Array und das zugehörige konjugiert-komplexe Rückwärts-Array kombiniert werden, um besseres Verhalten zu erzielen. Dies wird dadurch erreicht, daß der Mittelwert von Vor- und Rückwärtskovarianzmatrizen gebildet und anschließend der MUSIC-Algorithmus verwendet wird. Außerdem wird eine asymptotische Analyse von Mittelwert und Varianz des Nullraum-Spektralschätzers vorgestellt. Es wird gezeigt, daß bei unkorrelierter Umgebung Mittelwert und Varianz genau um den Faktor zwei gegenüber dem herkömmlichen Verfahren reduziert werden können. Die Ausdrücke für den Schätzfehler werden darüberhinaus verwendet, um eine Auflösungsgrenze für zwei unkorrelierte, im weißen Rauschen eingebettete ebene Wellen gleicher Leistung anzugeben. Dieses Ergebnis wird mit der Grenze, die das herkömmliche Verfahren für gleiche Eingangssignale liefert, verglichen.

Résumé. Cet article propose une technique à haute résolution améliorée qui incorpore la séquence avant (originale) avec sa version conjugué complexe arrière pour atteindre des performances supérieures. Ceci est réalisé, à son tour, en calculant la moyenne de la matrice de covariance avant et arrière et en l'utilisant en conjonction avec la technique MUSIC. Cet article présente également l'analyse asymptotique en termes des approximations d'ordre un de la moyenne et de la variance de l'estimateur spectral sans effet. Il est montré ici, que dans une scène sans corrélation, le biais et la variance correspondant à cette méthode sont exactement la moitié de ceux associés avec la méthode conventionnelle. En plus, les expressions de biais de la nouvelle méthode sont utilisées pour obtenir un seuil pour deux ondes planes non corrélées et à énergie égale dans du bruit blanc et le résultat est comparé à celui obtenu par la méthode conventionnelle pour la même scène de source.

Keywords. Underwater signal processing, detection and estimation of multiple signals.

1. Introduction

The eigenstructure-based high-resolution techniques for estimating the arrival angles of multiple plane wave signals have been of great interest since the well-known works of Pisarenko [14], Schmidt [16] and others [2-4, 7, 9, 11, 17]. These methods, in general, utilize certain eigenstructure properties resulting from the special structure of the sensor array output covariance matrix for planar wavefronts [16] to generate

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spectral peaks (or equivalently spectral nulls) along the actual directions of arrival whenever the sources are at most partially correlated.

When the exact ensemble average of the array output covariances are used, all these methods result in unbiased values (i.e., zero for the null spectrum) along the true arrival angles, irrespective of signal-to-noise ratios (SNRs) and angular separations of sources. However, when these covariances are estimated from a finite number of independent snapshots, these techniques exhibit deviations from their ensemble average values. These deviations depend on the specific scheme under consideration together with the SNRs and other signal and array specifications. All these taken together determine the resolution capacity of the technique under consideration.

This paper proposes an improved high-resolution technique without compromising on the size of the original array. In addition to the forward (original) array, this new scheme makes use of a complex conjugated backward version of the original one to achieve superior performance. This is in turn realized by averaging the forward covariance matrix and the backward one, and using this averaged matrix in conjunction with the MUSIC estimator. A detailed performance analysis of this new scheme, when covariances estimated from a finite sample size are used in place of their ensemble averages, is also presented along with first-order approximations of the mean and variance of the null spectrum estimator.

The organization of this paper is as follows: For clarity of presentation, the conventional scheme is summarized and the proposed one is described in Section 2. Using results derived in Appendix A, Section 3 presents the first-order approximations of the mean and variance of the null spectrum estimator corresponding to this new scheme and the conventional MUSIC scheme. The bias expressions are then used to obtain a resolution threshold for two uncorrelated, equipowered plane wave sources in white noise; and this result is compared to the resolution threshold in the conventional case [8] for the same scene.

2. Problem formulation

Let a uniform array consisting of M sensors receive signals from K narrowband sources $u_1(t)$, $u_2(t), \ldots, u_K(t)$, which are at most partially correlated. Furthermore, the respective arrival angles are assumed to be $\theta_1, \theta_2, \ldots, \theta_K$ with respect to the line of the array. Using complex envelope representation, the received signal $x_i(t)$ at the ith sensor can be expressed as [12]

$$x_i(t) = \sum_{k=1}^{K} u_k(t)\, e^{-j\pi(i-1)\cos\theta_k} + n_i(t). \tag{1}$$

Here the inter-element distance is taken to be half the wavelength common to all signals and $n_i(t)$ represents the additive noise at the ith sensor. It is assumed that the signals and noises are stationary, zero-mean circular Gaussian[1] independent random processes, and further, the noises are assumed to be independent and identical between themselves with common variance σ^2. Rewriting (1) in common vector

[1] A complex random vector z is defined to be circular Gaussian if its real part x and imaginary part y are jointly Gaussian and their joint covariance matrix has the form [5, 10]

$$E\left[\begin{bmatrix} x \\ y \end{bmatrix}[x^{\mathrm{T}} \quad y^{\mathrm{T}}]\right] = \frac{1}{2}\begin{bmatrix} V & -W \\ W & V \end{bmatrix}$$

where $z = x + jy$. When z has zero mean, its covariance matrix is given by $E[zz^{\dagger}] \triangleq E[(x+jy)(x^{\mathrm{T}} - jy^{\mathrm{T}})] = V + jW$. Clearly, $E(zz^{\mathrm{T}}) = 0$. Here onwards A^{T}, $A^{*\mathrm{T}} \triangleq A^{\dagger}$ stand for the transpose and the complex conjugate transpose of A, respectively.

notation and with $\omega_k = \pi \cos \theta_k$, $k = 1, 2, \ldots, K$, we have

$$x^{\mathrm{f}}(t) = \sqrt{M} \sum_{k=1}^{K} u_k(t) a(\omega_k) + n(t), \tag{2}$$

where $x^{\mathrm{f}}(t)$ is the $M \times 1$ vector

$$x^{\mathrm{f}}(t) = [x_1(t), x_2(t), \ldots, x_M(t)]^{\mathrm{T}}, \tag{3}$$

and $a(\omega_k)$ is the normalized direction vector associated with the arrival angle θ_k, i.e.,

$$a(\omega_k) = \frac{1}{\sqrt{M}} [1, \mathrm{e}^{-\mathrm{j}\omega_k}, \mathrm{e}^{-\mathrm{j}2\omega_k}, \ldots, \mathrm{e}^{-\mathrm{j}(M-1)\omega_k}]^{\mathrm{T}}. \tag{4}$$

The array output vector $x^{\mathrm{f}}(t)$ can further be rewritten as

$$x^{\mathrm{f}}(t) = Au(t) + n(t), \tag{5}$$

where

$$u(t) = [u_1(t), u_2(t), \ldots, u_K(t)]^{\mathrm{T}}, \qquad n(t) = [n_1(t), n_2(t), \ldots, n_M(t)]^{\mathrm{T}}, \tag{5a}$$

and

$$A = \sqrt{M}[a(\omega_1), a(\omega_2), \ldots, a(\omega_K)]. \tag{5b}$$

Here A is an $M \times K$ matrix with Vandermonde-structured columns $(M > K)$ of rank K. From our assumptions it follows that the (forward) array output covariance matrix has the form

$$R^{\mathrm{f}} \triangleq E[x^{\mathrm{f}}(t) x^{\mathrm{f}\dagger}(t)] = A R_u^{\mathrm{f}} A^{\dagger} + \sigma^2 I, \tag{6}$$

where

$$R_u^{\mathrm{f}} \triangleq E[u(t) u^{\dagger}(t)] \tag{6a}$$

represents the source covariance matrix which remains as nonsingular so long as the sources are at most partially correlated. In that case $A R_u^{\mathrm{f}} A^{\dagger}$ is also of rank K and hence, if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_M$ and $\beta_1, \beta_2, \ldots, \beta_M$ are the eigenvalues and the corresponding eigenvectors of R^{f} respectively, i.e.,

$$R^{\mathrm{f}} = \sum_{l=1}^{M} \lambda_l \beta_l \beta_l^{\dagger}, \tag{7}$$

then the above rank property implies that $\lambda_{K+1} = \lambda_{K+2} = \cdots = \lambda_M = \sigma^2$ and the eigenvectors corresponding to these equal eigenvalues are orthogonal to the direction vectors associated with the true arrivals, i.e., $\beta_i^{\dagger} a(\omega_k) = 0$, $i = K+1, K+2, \ldots, M$, $k = 1, 2, \ldots, K$. Hence the K nulls of $Q(\omega)$ given by

$$Q(\omega) = \sum_{k=K+1}^{M} |\beta_k^{\dagger} a(\omega)|^2 = 1 - \sum_{k=1}^{K} |\beta_k^{\dagger} a(\omega)|^2 \tag{7a}$$

correspond to the actual directions of arrival [16].

To improve the performance of the above estimator, in addition to the forward (original) array, we propose to make use of the complex conjugated backward array of the original one. Let $x^{\mathrm{b}}(t)$ denote the complex conjugate of the backward array, i.e.,

$$x^{\mathrm{b}}(t) = [x_M^*(t), x_{M-1}^*(t), \ldots, x_1^*(t)]^{\mathrm{T}} = Au^{\mathrm{b}}(t) + n^{\mathrm{b}}(t), \tag{8}$$

where

$$u^{b}(t) \triangleq [u_1^{b}(t), u_2^{b}(t), \ldots, u_K^{b}(t)]^{T}, \qquad u_k^{b}(t) = u_k^{*}(t)\, e^{j(M-1)\omega_k}, \tag{8a}$$

and

$$n^{b}(t) = [n_M^{*}(t), n_{M-1}^{*}(t), \ldots, n_1^{*}(t)]^{T}. \tag{8b}$$

It follows that the backward array output covariance matrix has the form

$$R^{b} = E[x^{b}(t)x^{b\dagger}(t)] = AR_u^{b}A^{\dagger} + \sigma^2 I, \tag{9}$$

where

$$R_u^{b} = E[u^{b}(t)u^{b\dagger}(t)]. \tag{9a}$$

Averaging the forward covariance matrix and the backward one together, we define the forward/backward (f/b) covariance matrix as

$$\tilde{R} \triangleq \tfrac{1}{2}(R^{f} + R^{b}) = A\tilde{R}_u A^{\dagger} + \sigma^2 I, \tag{10}$$

where

$$\tilde{R}_u \triangleq \tfrac{1}{2}(R_u^{f} + R_u^{b}). \tag{10a}$$

It is easy to show that whenever the rank of R_u^{f} is at least $K-1$, the same is true for R_u^{b} and moreover in that case $\tilde{R}_u$ is of rank K [12, 13]. Hence the eigenvalues of $\tilde{R}$ satisfy $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_K > \tilde{\lambda}_{K+1} = \tilde{\lambda}_{k+2} = \cdots = \tilde{\lambda}_M = \sigma^2$. Consequently, as in (7a) the K nulls of $\tilde{Q}(\omega)$ given by

$$\tilde{Q}(\omega) = \sum_{k=K+1}^{M} |\tilde{\beta}_k^{\dagger} a(\omega)|^2 = 1 - \sum_{k=1}^{K} |\tilde{\beta}_k^{\dagger} a(\omega)|^2 \tag{11}$$

will correspond to the actual directions of arrival. Here $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_M$ are the eigenvectors of $\tilde{R}$ corresponding to the eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_M$; i.e.,

$$\tilde{R} = \sum_{l=1}^{M} \tilde{\lambda}_l \tilde{\beta}_l \tilde{\beta}_l^{\dagger}.$$

Notice that the above lower bound on the rank condition (rank $R_u^{f} = K-1$) physically corresponds to one coherent arrival among the K sources; and further the full rank property for the equivalent source covariance matrix $\tilde{R}_u$ in that case implies that the coherent source has been essentially decorrelated by the simultaneous use of the forward and backward arr[illegible]. However, the main advantages of this modified scheme do not become apparent until its performance [illegible]uation is completed in the more realistic "data-only known" case.

So far we have assumed that an ensemble average of the array output covariances are available. Usually, these exact averages are unknown and they are estimated from the array output data. Often this is carried out for the unknowns of interest using the maximum likelihood procedure. For zero-mean M-variate (circular) Gaussian data $x^{f}(t_n)$, $n = 1, 2, \ldots, N$ in (5), with unknown $M \times M$ covariance matrix R^{f}, the maximum likelihood (ML) estimate S^{f} of the covariance matrix is given, referring to [1], as

$$S^{f} = \frac{1}{N} \sum_{n=1}^{N} x^{f}(t_n)x^{f\dagger}(t_n).$$

Using the invariant property of the maximum likelihood procedure, the corresponding estimates S^{b} and $\tilde{S}$ for the unknown matrices R^{b} and $\tilde{R}$ can be constructed from S^{f} by the same rule that is used in constructing R^{b} and $\tilde{R}$, respectively, from R^{f}, i.e.,

$$\tilde{S} = \tfrac{1}{2}(S^{f} + S^{b}),$$

with

$$S^{\mathrm{b}} = \frac{1}{N} \sum_{n=1}^{N} x^{\mathrm{b}}(t_n) x^{\mathrm{b}\dagger}(t_n).$$

In what follows we study the statistical properties of these estimated covariance matrices and their associated sample estimators for direction finding.

3. Performance analysis

3.1. Main results

This section examines the statistical behavior of the proposed forward/backward scheme and derives expressions for the bias and the resolution threshold for two equipowered uncorrelated sources. These results are substantiated by simulation studies and comparisons made with similar results obtained in the conventional case [8]. Toward this purpose, consider the eigen-representation

$$\tilde{S} = \tilde{E}\tilde{L}\tilde{E}^{\dagger} \tag{12}$$

for the ML estimate of the f/b matrix $\tilde{R}$, where

$\tilde{E} = [\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_K, \tilde{e}_{K+1}, \ldots, \tilde{e}_M]$, $\tilde{L} = \mathrm{diag}$[2] $[\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_K, \tilde{l}_{K+1}, \ldots, \tilde{l}_M]$, and $\tilde{E}\tilde{E}^{\dagger} = I_M$,

which satisfies $\tilde{e}_{ii} \geqslant 0$, $i = 1, 2, \ldots, M$ for uniqueness. Here the normalized vectors $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_K$ are the ML estimates of the eigenvectors $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_K$ of $\tilde{R}$ respectively [5]. Similarly, $\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_K$ are the ML estimates of the K largest and distinct eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_K$, and the mean of $\tilde{l}_{K+1}, \ldots, \tilde{l}_M$ is the sample estimate of the repeating lowest eigenvalue σ^2 of $\tilde{R}$. Following (11), the sample direction estimator can be written as

$$\hat{Q}(\omega) = \sum_{k=K+1}^{M} |\tilde{e}_i^{\dagger} a(\omega)|^2 = 1 - \sum_{k=1}^{K} |\tilde{e}_i^{\dagger} a(\omega)|^2. \tag{13}$$

The asymptotic distribution of the estimates of the eigenvalues and eigenvectors associated with the distinct eigenvalues of $\tilde{R}$ is derived in (A.35)-(A.36) of Appendix A. Corresponding results for the conventional scheme in (7a) can be readily evaluated as special cases of this general result. It is also shown that that the estimated eigenvalues and a specific set of corresponding unnormalized eigenvectors are asymptotically (in the sense of large N) jointly Gaussian with means and covariances as derived there (see (A.37) and (A.38)). Further, after proper renormalization and using an exact relationship developed in Appendix B among the different sets of eigenvectors, it is shown in Appendix A that (see A.49)

$$\begin{aligned} E[\hat{Q}(\omega)] = \tilde{Q}(\omega) + \frac{1}{N} \sum_{i=1}^{K} \Bigg[& \sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{\tilde{\Gamma}_{iikk}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)^2} |\tilde{\beta}_i^{\dagger} a(\omega)|^2 \\ & - \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq i}}^{M} \frac{\tilde{\Gamma}_{iikl}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_i - \tilde{\lambda}_l)} a^{\dagger}(\omega) \tilde{\beta}_k \tilde{\beta}_l^{\dagger} a(\omega) \Bigg] + \mathrm{o}(1/N^2), \end{aligned} \tag{14}$$

where from (A.18)

$$\tilde{\Gamma}_{iklj} \triangleq \tfrac{1}{4}[\tilde{\beta}_i^{\dagger} R^{\mathrm{f}} \tilde{\beta}_k \tilde{\beta}_l^{\dagger} R^{\mathrm{f}} \tilde{\beta}_j + \tilde{\beta}_i^{\dagger} R^{\mathrm{b}} \tilde{\beta}_k \tilde{\beta}_l^{\dagger} R^{\mathrm{b}} \tilde{\beta}_l + \tilde{\beta}_i^{\dagger} R^{\mathrm{f}} \tilde{\gamma}_l \tilde{\gamma}_j^{\dagger} R^{\mathrm{b}} \tilde{\beta}_k + \tilde{\beta}_i^{\dagger} R^{\mathrm{b}} \tilde{\gamma}_l \tilde{\gamma}_j^{\dagger} R^{\mathrm{f}} \tilde{\beta}_k]. \tag{15}$$

[2] $\mathrm{diag}[\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_M]$ represents an $M \times M$ diagonal matrix with diagonal entries $\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_M$, respectively.

In particular for the conventional MUSIC case with (A.45) in (14), after some simplifications it reduces to

$$E[\hat{Q}(\omega)] = Q(\omega) + \frac{1}{N}\sum_{i=1}^{K}\left[\sum_{\substack{k=1\\k\neq i}}^{M}\frac{\lambda_i\lambda_k}{(\lambda_i-\lambda_k)^2}\{|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2 - |\boldsymbol{\beta}_k^{\dagger}\boldsymbol{a}(\omega)|^2\}\right] + \mathrm{o}(1/N^2)$$

$$= Q(\omega) + \frac{1}{N}\sum_{i=1}^{K}\frac{\lambda_i\sigma^2}{(\lambda_i-\sigma^2)^2}[(M-K)|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2 - Q(\omega)] + \mathrm{o}(1/N^2), \tag{16}$$

where λ_i, $\boldsymbol{\beta}_i$, $i = 1, 2, \ldots, M$, are as defined in (7) and $Q(\omega)$ is given by (7a).

Similarly, from (A.52)

$$\mathrm{Var}(\hat{Q}(\omega)) \triangleq \tilde{\sigma}^2(\omega)$$

$$= \frac{2}{N}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M}\frac{\mathrm{Re}[(\tilde{\Gamma}_{klji}\boldsymbol{a}^{\dagger}(\omega)\tilde{\boldsymbol{\beta}}_j\tilde{\boldsymbol{\beta}}_l^{\dagger}\boldsymbol{a}(\omega) + \tilde{\Gamma}_{kjli}\boldsymbol{a}^{\dagger}(\omega)\tilde{\boldsymbol{\beta}}_l\tilde{\boldsymbol{\beta}}_j^{\dagger}\boldsymbol{a}(\omega))\boldsymbol{a}^{\dagger}(\omega)\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{a}(\omega)]}{(\tilde{\lambda}_i-\tilde{\lambda}_k)(\tilde{\lambda}_j-\tilde{\lambda}_l)}$$

$$+ \mathrm{o}(1/N^2), \tag{17}$$

which again for the conventional MUSIC case reduces to

$$\mathrm{Var}(\hat{Q}(\omega)) \triangleq \sigma^2(\omega)$$

$$= \frac{2}{N}\sum_{i=1}^{K}\left[\sum_{\substack{k=1\\k\neq i}}^{M}\frac{\lambda_i\lambda_k}{(\lambda_i-\lambda_k)^2}|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2|\boldsymbol{\beta}_k^{\dagger}\boldsymbol{a}(\omega)|^2 - \sum_{\substack{j=1\\j\neq i}}^{K}\frac{\lambda_i\lambda_j}{(\lambda_i-\lambda_j)^2}|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2|\boldsymbol{\beta}_j^{\dagger}\boldsymbol{a}(\omega)|^2\right] + \mathrm{o}(1/N^2)$$

$$= \frac{2}{N}Q(\omega)\sum_{i=1}^{K}\frac{\lambda_i\sigma^2}{(\lambda_i-\sigma^2)^2}|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2 + \mathrm{o}(1/N^2). \tag{18}$$

Since along the actual arrival angles, $Q(\omega_k) = 0$, $k = 1, 2, \ldots, K$, (18) allows us to conclude that within the above approximation,

$$\sigma^2(\omega_k) \cong 0, \quad k = 1, 2, \ldots, K, \tag{19}$$

i.e., in all multiple-target situations, where the conventional MUSIC scheme is applicable, the variance of the sample estimator for (7a) along the true arrival angles is zero within a first-order approximation. It is easily verified that this result agrees with those of Kaveh et al., for a two-source case (see equation (30) in [8]). An algebraic manipulation shows that their $\mathrm{Var}(\hat{D}(\omega_k)) = 0$, agreeing with (19) here.

In particular, when all signals are uncorrelated with one another, that leads to some interesting results. In that case, the backward covariance matrix in (9) is identical to the forward covariance matrix in (6), i.e., $\boldsymbol{R}^{\mathrm{f}} = \boldsymbol{R}^{\mathrm{b}} = \tilde{\boldsymbol{R}}$ implying equality of all eigenvalues and eigenvectors for these matrices. In that case, from (A.20),

$$\tilde{\Gamma}_{iklj} = \tfrac{1}{2}\lambda_i\lambda_j(\delta_{ik}\delta_{lj} + \boldsymbol{\beta}_i^{\dagger}\boldsymbol{\gamma}_l\boldsymbol{\gamma}_j^{\dagger}\boldsymbol{\beta}_k). \tag{20}$$

Let $\tilde{\eta}(\omega)$ and $\eta(\omega)$ denote the bias in the f/b and the conventional null spectrum respectively. From (14)-(18) and (20),

$$\tilde{\eta}(\omega) = E[\hat{Q}(\omega)] - \tilde{Q}(\omega)$$

$$= \frac{1}{2N}\sum_{i=1}^{K}\frac{\lambda_i\sigma^2}{(\lambda_i-\sigma^2)^2}[(M-K)|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2 - \tilde{Q}(\omega)] + o(1/N^2) = \tfrac{1}{2}\eta(\omega) + o(1/N^2) \tag{21}$$

and

$$\tilde{\sigma}^2(\omega) = \frac{2}{N}Q(\omega)\sum_{i=1}^{K}\frac{\lambda_i\sigma^2}{(\lambda_i-\sigma^2)^2}|\boldsymbol{\beta}_i^{\dagger}\boldsymbol{a}(\omega)|^2 + o(1/N^2) = \sigma^2(\omega). \tag{22}$$

Notice that once again $\tilde{\sigma}^2(\omega_k) = 0$, $k = 1, 2, \ldots, K$. Though there is no significant advantage in using $\tilde{\boldsymbol{R}}$ instead of $\boldsymbol{R}^{\mathrm{f}}$ when the exact covariances are known (except when there is a coherent source), this is no longer true when these covariances are estimated from the data. Then, in an uncorrelated source scene, only half the number of samples are required to maintain the same performance in terms of bias compared with the conventional scheme while maintaining the same variance. Similar improvements in performance can also be deduced for a partially correlated source scene $(0 < |\rho_{ij}| < 1)$. To see this, let

$$\rho_{ij} = \frac{E[u_i(t)u_j^*(t)]}{(E[|u_i(t)|^2]E[|u_j(t)|^2])^{1/2}} \triangleq |\rho_{ij}|\,\mathrm{e}^{\mathrm{j}\phi_{ij}}, \quad i, j = 1, 2, \ldots, K$$

represent the correlation coefficient between the signals $u_i(t)$ and $u_j(t)$. Then in the case of the f/b scheme, the equivalent correlation coefficient $\tilde{\rho}_{ij}$ appearing in (10a) for the same pair of signals can be shown to be [12]

$$\tilde{\rho}_{ij} = |\rho_{ij}|\,\mathrm{e}^{-\mathrm{j}(M-1)\omega_{ij}}\cos((M-1)\omega_{ij} + \phi_{ij}), \quad \omega_{ij} = (\omega_j - \omega_i)/2.$$

Clearly, $|\tilde{\rho}_{ij}| \leq |\rho_{ij}|$; or stated in words, the f/b scheme has in effect decorrelated the signals beyond their original correlation level. Since uncorrelated signals have superior performance in this new scheme, any amount of decorrelation will essentially lead to improved performance.

The general expressions for the bias and variance in (14) and (17) can be used to determine the required sample size for a certain performance level or a useful resolution criteria. Though the general cases are intractable, an analysis is possible for two uncorrelated sources. As shown in the next section, the performance of the proposed scheme can be evaluated in terms of a resolution threshold for a two source scene.

3.2. *Two-source case*

In this section we will assume that the two sources present in the scene are uncorrelated with each other. In that case, from (21) and (22) with $K = 2$, the f/b scheme is uniformly superior to the conventional one in terms of the bias of the estimator by a factor of two. This conclusion is also supported by simulation results presented in Fig. 1 with details as indicated there.

The deviation of $\eta(\omega_i)$ and $\tilde{\eta}(\omega_i)$ from zero—their nominal value—suggests the loss in resolution for the respective estimators. Since the estimators have zero variance along the two arrival angles in both cases, for a fixed number of samples a threshold in terms of SNR exists, below which the two nulls corresponding to the true arrival angles are no longer identifiable. This has led to the definition of the resolution threshold for two closely spaced sources as that value of SNR at which [8]

$$E[\hat{Q}(\omega_1)] = E[\hat{Q}(\omega_2)] = E[\hat{Q}((\omega_1 + \omega_2)/2)], \tag{23}$$

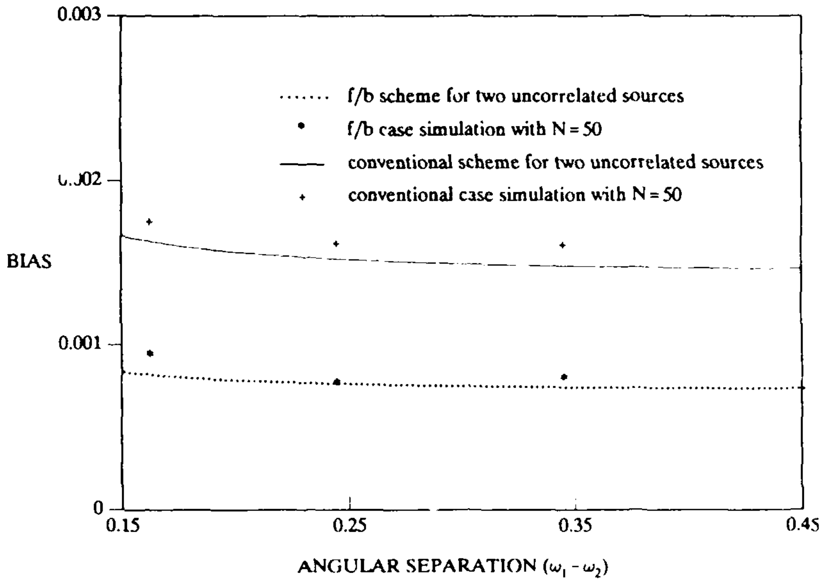

Fig. 1. Bias at one of the arrival angles vs. angular separation for two equipowered sources in an uncorrelated scene. A seven element array is used to collect signals in all these cases. Input SNR is taken to be 10 dB.

whenever $\mathrm{Var}(\hat{Q}(\omega_1)) \approx \mathrm{Var}(\hat{Q}(\omega_2)) \approx 0$. To simplify (23), notice that in the case of two uncorrelated sources with unit power, the eigenvalues and eigenvectors of the signal subspace has the form [12],

$$\mu_i = M(1 \pm |\rho_s|), \tag{24}$$

and

$$\boldsymbol{\beta}_i = \begin{cases} (\boldsymbol{u}_1 \pm \boldsymbol{u}_2)/\sqrt{2(1 \pm \mathrm{Si}(M\omega_d))}, & \mathrm{Si}(M\omega_d) > 0, \\ (\boldsymbol{u}_1 \mp \boldsymbol{u}_2)/\sqrt{2(1 \mp \mathrm{Si}(M\omega_d))}, & \text{otherwise}, \end{cases} \quad ; \quad i = 1, 2, \tag{25}$$

where

$$\boldsymbol{u}_1 = \mathrm{e}^{\mathrm{j}(M-1)\omega_1/2}\boldsymbol{a}(\omega_1), \qquad \boldsymbol{u}_2 = \mathrm{e}^{\mathrm{j}(M-1)\omega_2/2}\boldsymbol{a}(\omega_2), \tag{26}$$

with

$$\rho_s = \boldsymbol{a}^{\dagger}(\omega_1)\boldsymbol{a}(\omega_2) = \mathrm{e}^{\mathrm{j}(M-1)\omega_d}\mathrm{Si}(M\omega_d), \qquad \mathrm{Si}(M\omega_d) = \frac{\sin M\omega_d}{M \sin \omega_d}; \qquad \omega_d \triangleq \frac{(\omega_1 - \omega_2)}{2}. \tag{27}$$

Defining $\Delta^2 = (M^2\omega_d^2)/3$, we can find several inner products that are valid for $\mathrm{Si}(M\omega_d) > 0$. In particular,

$$|\boldsymbol{\beta}_1^{\dagger}\boldsymbol{a}(\omega_i)|^2 = \frac{1 + \mathrm{Si}(M\omega_d)}{2} \approx 1 - \tfrac{1}{4}\Delta^2 + \tfrac{3}{80}\Delta^4, \tag{28}$$

$$|\boldsymbol{\beta}_2^{\dagger}\boldsymbol{a}(\omega_i)|^2 = \frac{1 - \mathrm{Si}(M\omega_d)}{2} \approx \tfrac{1}{4}\Delta^2 - \tfrac{3}{80}\Delta^4, \tag{29}$$

for $i = 1, 2$.

With the midangle $\omega_m = (\omega_1 + \omega_2)/2$, we also have

$$|\boldsymbol{\beta}_1^\dagger \boldsymbol{a}(\omega_m)|^2 = \frac{2(\mathrm{Si}(M\omega_d/2))^2}{1+\mathrm{Si}(M\omega_d)} \approx 1 - \tfrac{1}{80}\Delta^4 \tag{30}$$

and

$$|\boldsymbol{\beta}_2^\dagger \boldsymbol{a}(\omega_m)|^2 \approx 0. \tag{31}$$

Using these inner products, Kaveh et al. found the resolution threshold $\xi_T (= MP/\sigma^2$, array output SNR) for the conventional scheme to be [8]

$$\xi_T \approx \frac{1}{N}\left[\frac{20(M-2)}{\Delta^4}\left\{1+\left(1+\frac{N}{5(M-2)}\Delta^2\right)^{1/2}\right\}\right]. \tag{32}$$

The corresponding threshold $\tilde{\xi}_T$ for the proposed (f/b) scheme can also be found by using these norms. In that case from (23),

$$\tilde{\xi}_T \approx \frac{1}{N}\left[\frac{10(M-2)}{\Delta^4}\left\{1+\left(1+\frac{2N}{5(M-2)}\Delta^2\right)^{1/2}\right\}\right] \approx \xi_T/2. \tag{33}$$

This asymptotic analysis is also found to be in agreement with the results obtained by Monte Carlo simulations. A typical case study is reported in Table 1. Under the equality conditions in (23), the probability of resolution was found to be 0.3 in both cases there. This in turn implies that the above analysis should give an approximate threshold in terms of ξ for 0.3 probability of resolution. Comparisons are carried out in Fig. 2 using (32), (33) and simulation results from Table 1 for 0.3 probability of resolution. Figure 3 shows a similar comparison for another array length. In all these cases the close agreement between the theory and simulation results is clearly evident.

Table 1
Resolution threshold and probability of resolution versus angular separation for two equipowered sources in uncorrelated scenes

Angles of arrival		Angular separation	Uncorrelated		Uncorrelated (f/b)	
θ_1	θ_2	$2\omega_d$	SNR (dB)	Prob.	SNR (dB)	Prob.
40.00	43.00	0.1090	16	0.21	13	0.24
			17	0.36	14	0.38
			18	0.48	15	0.39
			19	0.70	16	0.58
			20	0.81	17	0.61
55.00	58.00	0.1372	12	0.20	9	0.21
			13	0.40	10	0.37
			14	0.50	11	0.45
			15	0.59	12	0.65
			16	0.83	13	0.67
120.00	124.00	0.1860	6	0.15	4	0.25
			7	0.24	5	0.38
			8	0.48	6	0.50
			9	0.50	7	0.57
			10	0.79	8	0.75

Number of sensors = 10, number of snapshots = 75, number of simulations = 100.

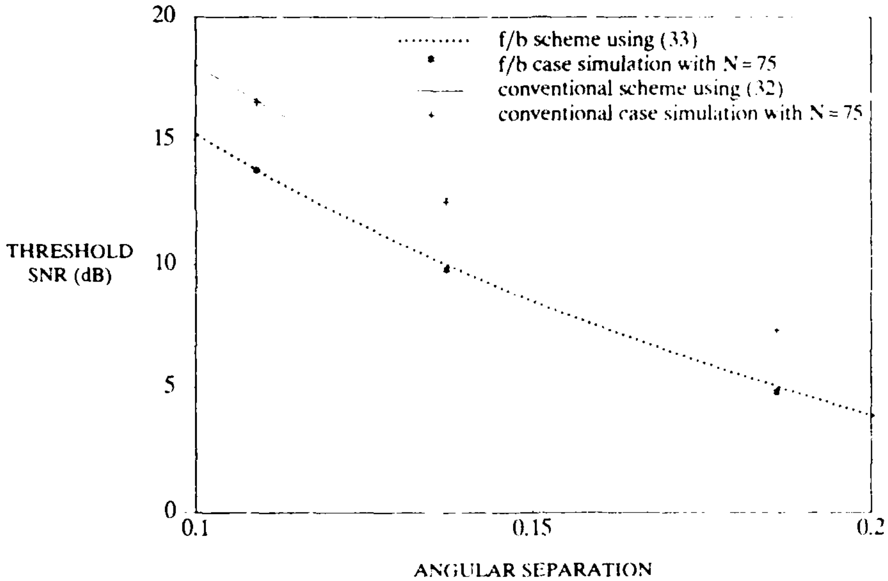

Fig. 2. Resolution threshold vs. angular separation for two equipowered sources in an uncorrelated scene. A ten element array is used to receive signals in both cases.

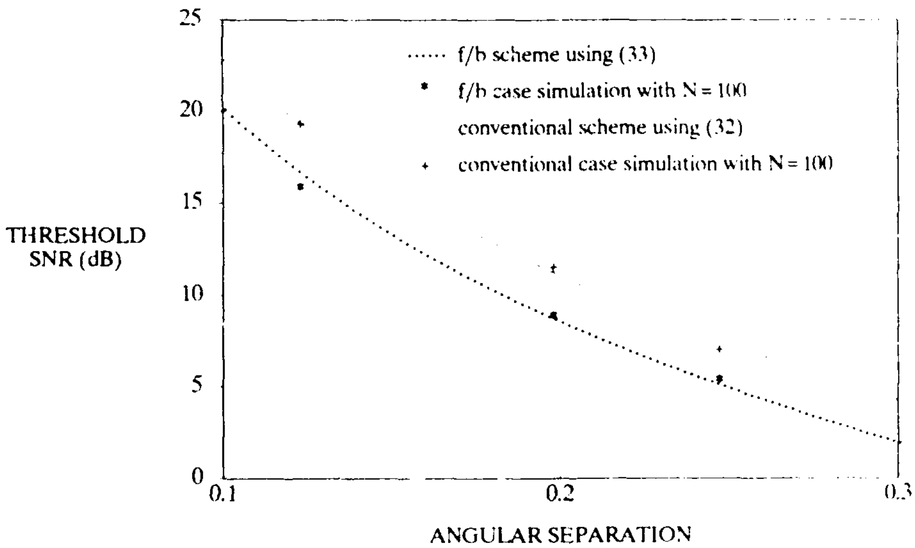

Fig. 3. Resolution threshold vs. angular separation for two equipowered sources in an uncorrelated scene. A seven element array is used to receive signals in both cases.

The above 0.3 probability of resolution can be explained by re-examining the arguments used in deriving the resolution thresholds (32) and (33). In fact, (23) has been justified by observing that $\mathrm{Var}(\hat{Q}(\omega_1)) = \mathrm{Var}(\hat{Q}(\omega_2)) = 0$. Although $\mathrm{Var}(\hat{Q}((\omega_1+\omega_2)/2))$ is equally important in that analysis, using (18) and (22) it is easy to see that these variances along the midangle have nonzero values. This implies that though ξ_1 and $\tilde{\xi}_1$ satisfy (23), in an actual set of trials the estimated mean value of $\hat{Q}((\omega_1+\omega_2)/2)$ will almost always be in the interval $(0, 2E[\hat{Q}((\omega_1+\omega_2)/2)])$ and clearly the resolution of the two nulls in $\hat{Q}(\omega)$ is possible only if this mean estimate lies in the upper half of the above interval. In the special case of a symmetrical density function for the mean value estimate along the midangle, this occurs with probability 0.5 and the observed range may be attributed to the skewed nature of the actual probability density function.

4. Conclusions

An improved high-resolution technique that incorporates the forward array together with its complex conjugated backward form for estimating the directions of arrival of multiple signals is described here. An asymptotic analysis together with expressions for mean and variance of this proposed f/b scheme is presented first; and the conventional MUSIC scheme is then derived as a special case of this general analysis. In particular, when all signals are uncorrelated, the bias of the null spectrum estimator for the f/b scheme is found to be half of that in the conventional case. Further, a resolution threshold, which depends on the relative angular separation, number of sensors, number of snapshots and signal-to-noise ratios, for two uncorrelated, equi-powered, closely spaced signals is derived here for this new scheme and this is compared to a similar result obtained in the conventional case [8]. From these comparisons, to detect two arrival angles under identical conditions one needs approximately half the number of snapshots with respect to the conventional case. These conclusions are also seen to closely agree with the results obtained from Monte Carlo simulation studies. For this scheme, similar computations can be carried out in a two coherent source scene in order to obtain the corresponding threshold [12, 13].

Appendix A. Asymptotic distribution of the sample eigenvalues and eigenvectors corresponding to distinct eigenvalues of $\tilde{\boldsymbol{R}}$

With symbols as defined in the text, $\tilde{\boldsymbol{S}}$ representing the ML estimate of the forward/backward (f/b) covariance matrix $\tilde{\boldsymbol{R}}$, we have

$$\tilde{\boldsymbol{R}} = \tfrac{1}{2}\{\boldsymbol{R}^{\mathrm{f}} + \boldsymbol{R}^{\mathrm{b}}\} \triangleq \tilde{\boldsymbol{B}}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{B}}^{\dagger}, \tag{A.1}$$

$$\tilde{\boldsymbol{S}} = \frac{1}{2}\left[\frac{1}{N}\sum_{n=1}^{N}\{\boldsymbol{x}^{\mathrm{f}}(n)(\boldsymbol{x}^{\mathrm{f}}(n))^{\dagger} + \boldsymbol{x}^{\mathrm{b}}(n)(\boldsymbol{x}^{\mathrm{b}}(n))^{\dagger}\}\right] \triangleq \hat{\boldsymbol{E}}\tilde{\boldsymbol{L}}\hat{\boldsymbol{E}}^{\dagger}, \tag{A.2}$$

where

$$E[\tilde{\boldsymbol{S}}] = \tilde{\boldsymbol{R}}, \qquad \tilde{\boldsymbol{B}} = [\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2, \ldots, \tilde{\boldsymbol{\beta}}_M], \qquad \hat{\boldsymbol{E}} = [\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \ldots, \hat{\boldsymbol{e}}_M],$$

$$\tilde{\boldsymbol{\Lambda}} = \mathrm{diag}[\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_K, \sigma^2, \ldots, \sigma^2], \qquad \tilde{\boldsymbol{L}} = \mathrm{diag}[\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_K, \tilde{l}_{K+1}, \ldots, \tilde{l}_M],$$

$$\tilde{\boldsymbol{B}}\tilde{\boldsymbol{B}}^{\dagger} = \hat{\boldsymbol{E}}\hat{\boldsymbol{E}}^{\dagger} = \boldsymbol{I}_M,$$

and $\tilde{\boldsymbol{B}}, \hat{\boldsymbol{E}}$ satisfies $\tilde{\boldsymbol{\beta}}_{ii}, \hat{e}_{ii} > 0$, $i = 1, 2, \ldots, M$ for uniqueness. As is well known, the eigenvectors are not unique, and let $\boldsymbol{C}$ represent yet another set of eigenvectors for $\tilde{\boldsymbol{S}}$, i.e.,

$$\tilde{\boldsymbol{S}} = \boldsymbol{C}\tilde{\boldsymbol{L}}\boldsymbol{C}^{\dagger}, \tag{A.3}$$

where

$$\boldsymbol{C} = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_M], \tag{A.4}$$

$$\boldsymbol{C}\boldsymbol{C}^{\dagger} = \boldsymbol{I}_M. \tag{A.5}$$

For reasons that will become apparent later, $\boldsymbol{C}$ is made unique here by requiring that all diagonal elements of $\boldsymbol{Y}$ given by

$$\boldsymbol{Y} = \tilde{\boldsymbol{B}}^{\dagger}\boldsymbol{C}, \tag{A.6}$$

are positive ($y_{ii} > 0$, $i = 1, 2, \ldots, M$). In what follows we first derive the asymptotic distribution of the set of sample eigenvectors and eigenvalues of $\tilde{\boldsymbol{S}}$ given by (A.3)-(A.6) and use this to analyze the performance of the sample directions of arrival estimator $\hat{Q}(\omega)$ in (13). This is made possible by noticing that although the estimated eigenvectors $\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \ldots, \tilde{\boldsymbol{e}}_K$, in (13) are structurally identical to their counterparts $\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2, \ldots, \tilde{\boldsymbol{\beta}}_K$, and in particular have $\tilde{e}_{ii} > 0$, nevertheless as shown in Appendix B, they are related to $\boldsymbol{c}_i$, $i = 1, 2, \ldots, K$, through a phase factor, i.e.,

$$\tilde{\boldsymbol{e}}_i = e^{j\phi_i}\boldsymbol{c}_i, \quad i = 1, 2, \ldots, K \tag{A.7}$$

and hence

$$\hat{Q}(\omega) = 1 - \sum_{i=1}^{K} |\tilde{\boldsymbol{e}}_i^{\dagger}\boldsymbol{a}(\omega)|^2 = 1 - \sum_{i=1}^{K} |\boldsymbol{c}_i^{\dagger}\boldsymbol{a}(\omega)|^2 = 1 - \sum_{i=1}^{K} y_i(\omega), \tag{A.8}$$

where

$$y_i(\omega) = |\boldsymbol{c}_i^{\dagger}\boldsymbol{a}(\omega)|^2. \tag{A.9}$$

Thus, the statistical properties of $\hat{Q}(\omega)$ can be completely specified by those of $\boldsymbol{c}_i$, $i = 1, 2, \ldots, K$, and toward this purpose, let

$$\boldsymbol{F} \triangleq \sqrt{N}(\tilde{\boldsymbol{L}} - \tilde{\boldsymbol{\Lambda}}), \tag{A.10}$$

$$\boldsymbol{G} = [\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_K, \ldots, \boldsymbol{g}_M] \triangleq \sqrt{N}(\boldsymbol{C} - \tilde{\boldsymbol{B}}), \tag{A.11}$$

$$\boldsymbol{T} \triangleq \tilde{\boldsymbol{B}}^{\dagger}\tilde{\boldsymbol{S}}\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}^{\dagger}\boldsymbol{C}\tilde{\boldsymbol{L}}\boldsymbol{C}^{\dagger}\tilde{\boldsymbol{B}} = \boldsymbol{Y}\tilde{\boldsymbol{L}}\boldsymbol{Y}^{\dagger}, \tag{A.12}$$

where $\boldsymbol{Y}$ is as defined in (A.6) with $y_{ii} > 0$, $i = 1, 2, \ldots, M$. Further, let

$$\begin{aligned}\boldsymbol{U} = \sqrt{N}(\boldsymbol{T} - \tilde{\boldsymbol{\Lambda}}) &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N}[\tfrac{1}{2}\tilde{\boldsymbol{B}}^{\dagger}(\boldsymbol{x}^{f}(n)(\boldsymbol{x}^{f}(n))^{\dagger} + \boldsymbol{x}^{b}(n)(\boldsymbol{x}^{b}(n))^{\dagger})\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{\Lambda}}]\\ &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N}[\tfrac{1}{2}(\boldsymbol{z}^{f}(n)(\boldsymbol{z}^{f}(n))^{\dagger} + \boldsymbol{z}^{b}(n)(\boldsymbol{z}^{b}(n))^{\dagger}) - \tilde{\boldsymbol{\Lambda}}],\end{aligned} \tag{A.13}$$

with

$$\boldsymbol{z}^{f}(n) = \tilde{\boldsymbol{B}}^{\dagger}\boldsymbol{x}^{f}(n) \sim N(0, \tilde{\boldsymbol{B}}^{\dagger}\boldsymbol{R}^{f}\tilde{\boldsymbol{B}}) \tag{A.14}$$

and

$$\boldsymbol{z}^{b}(n) = \tilde{\boldsymbol{B}}^{\dagger}\boldsymbol{x}^{b}(n) \sim N(0, \tilde{\boldsymbol{B}}^{\dagger}\boldsymbol{R}^{b}\tilde{\boldsymbol{B}}). \tag{A.15}$$

It is easily verified that these random vectors preserve the circular Gaussian property of the original data vectors. Again, from the independence of observations, asymptotically every entry in $\boldsymbol{U}$ is a sum of a large number of independent random variables. Using the multivariate central limit theorem [1], the limiting distribution of $\boldsymbol{U}$ tends to be normal with means and covariances given by

$$E[u_{ij}] = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} E[\tfrac{1}{2}(z_i^{f}(n)z_j^{f*}(n) + z_i^{b}(n)z_j^{b*}(n)) - \tilde{\lambda}_i\delta_{ij}] = 0. \tag{A.16}$$

(Here onwards whenever there is no confusion, we will suppress the time index n) since

$$E[\tfrac{1}{2}(z_i^{f}(z_j^{f})^* + z_i^{b}(z_j^{b})^*)] = \tilde{\boldsymbol{\beta}}_i^{\dagger}E[\tfrac{1}{2}\{\boldsymbol{x}^{f}(\boldsymbol{x}^{f})^{\dagger} + \boldsymbol{x}^{b}(\boldsymbol{x}^{b})^{\dagger}\}]\tilde{\boldsymbol{\beta}}_j = \tilde{\boldsymbol{\beta}}_i^{\dagger}E[\tilde{\boldsymbol{S}}]\tilde{\boldsymbol{\beta}}_j = \tilde{\boldsymbol{\beta}}_i^{\dagger}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{\beta}}_j = \tilde{\lambda}_i\delta_{ij}, \tag{A.17}$$

and

$$E[u_{ij}u_{kl}^*]=\frac{1}{N}\sum_{n=1}^{N}\left[\tfrac{1}{4}\{E(z_i^{\mathrm{f}}z_j^{\mathrm{f}*}z_k^{\mathrm{f}*}z_l^{\mathrm{f}})+E(z_i^{\mathrm{f}}z_j^{\mathrm{f}*}z_k^{\mathrm{b}*}z_l^{\mathrm{b}})+E(z_i^{\mathrm{b}}z_j^{\mathrm{b}*}z_k^{\mathrm{f}*}z_l^{\mathrm{f}})+E(z_i^{\mathrm{b}}z_j^{\mathrm{b}*}z_k^{\mathrm{b}*}z_l^{\mathrm{b}})\}\right]-\tilde{\lambda}_i\tilde{\lambda}_k\delta_{ij}\delta_{kl}.$$

Using the results[3] for fourth-order moments of jointly circular Gaussian random variables and after some algebraic manipulations, we have

$$E[u_{ij}u_{kl}^*]=\tfrac{1}{4}[\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\beta}}_j+\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\beta}}_j$$

$$+\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\gamma}}_l\tilde{\boldsymbol{\gamma}}_j^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\beta}}_k+\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\gamma}}_l\tilde{\boldsymbol{\gamma}}_j^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\beta}}_k]\triangleq\tilde{\Gamma}_{iklj}. \qquad (\mathrm{A.18})$$

In obtaining (A.18), we have made use of (A.17) and the fact that for circular Gaussian data

$$E[z_i^{\mathrm{f}}z_j^{\mathrm{b}*}]=\tilde{\boldsymbol{\beta}}_i^{\dagger}E[\boldsymbol{x}^{\mathrm{f}}\boldsymbol{x}^{\mathrm{b}\dagger}]\tilde{\boldsymbol{\beta}}_j=\tilde{\boldsymbol{\beta}}_i^{\dagger}E[\boldsymbol{x}^{\mathrm{f}}(\boldsymbol{x}^{\mathrm{f}})^{\mathrm{T}}]\boldsymbol{\gamma}_j^*=0. \qquad (\mathrm{A.19})$$

Here $\tilde{\boldsymbol{\gamma}}_j$ is the inverted $\tilde{\boldsymbol{\beta}}_j^*$ vector with $\tilde{\gamma}_{j,m}=\tilde{\beta}_{j,M-m+1}^*$. In an uncorrelated source scene, it is easy to show that $\tilde{\boldsymbol{\beta}}_i^{\dagger}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{\beta}}_j=\tilde{\boldsymbol{\gamma}}_i^{\dagger}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{\gamma}}_j$ for all i,j. Using results of Appendix B, it then follows that $\tilde{\boldsymbol{\gamma}}_i=\tilde{\boldsymbol{\beta}}_i\mathrm{e}^{\mathrm{j}\psi_i}$, $i=1,2,\dots,K$ and $[\tilde{\boldsymbol{\gamma}}_{K+1},\dots,\tilde{\boldsymbol{\gamma}}_M]=[\tilde{\boldsymbol{\beta}}_{K+1},\dots,\tilde{\boldsymbol{\beta}}_M]\boldsymbol{V}$, where $\boldsymbol{V}$ is an $(M-K)\times(M-K)$ unitary matrix. In that case, this together with the identity $\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{a}(\omega_k)=0$ for $i=K+1,K+2,\dots,M$ and $k=1,2,\dots,K$ simplifies the above expression into

$$\tilde{\Gamma}_{iklj}=\tfrac{1}{2}\tilde{\lambda}_i\tilde{\lambda}_j(\delta_{ik}\delta_{jl}+\tilde{\boldsymbol{\beta}}_i^{\dagger}\tilde{\boldsymbol{\gamma}}_l\tilde{\boldsymbol{\gamma}}_j^{\dagger}\tilde{\boldsymbol{\beta}}_k). \qquad (\mathrm{A.20})$$

Proceeding as in (A.18) we also have

$$E[u_{ij}u_{kl}]=\tfrac{1}{4}[\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\beta}}_l\tilde{\boldsymbol{\beta}}_k^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\beta}}_j+\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\beta}}_l\tilde{\boldsymbol{\beta}}_k^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\beta}}_j$$

$$+\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\gamma}}_k\tilde{\boldsymbol{\gamma}}_j^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\beta}}_l+\tilde{\boldsymbol{\beta}}_i^{\dagger}\boldsymbol{R}^{\mathrm{b}}\tilde{\boldsymbol{\gamma}}_k\tilde{\boldsymbol{\gamma}}_j^{\dagger}\boldsymbol{R}^{\mathrm{f}}\tilde{\boldsymbol{\beta}}_l]\triangleq\tilde{\Gamma}_{ilkj}. \qquad (\mathrm{A.21})$$

Using (A.12) together with (A.13) and (A.10) we have

$$\boldsymbol{T}=\tilde{\boldsymbol{\Lambda}}+\frac{1}{\sqrt{N}}\boldsymbol{U}=\boldsymbol{Y}\tilde{\boldsymbol{L}}\boldsymbol{Y}^{\dagger}=\boldsymbol{Y}\left(\tilde{\boldsymbol{\Lambda}}+\frac{1}{\sqrt{N}}\boldsymbol{F}\right)\boldsymbol{Y}^{\dagger},$$

which gives the identity

$$\tilde{\boldsymbol{\Lambda}}+\frac{1}{\sqrt{N}}\boldsymbol{U}=\boldsymbol{Y}\left(\tilde{\boldsymbol{\Lambda}}+\frac{1}{\sqrt{N}}\boldsymbol{F}\right)\boldsymbol{Y}^{\dagger}. \qquad (\mathrm{A.22})$$

To determine the asymptotic properties of the sample estimates corresponding to the distinct eigenvalues $\tilde{\lambda}_1,\dots,\tilde{\lambda}_K$ of $\tilde{\boldsymbol{R}}$, following [1,6], we partition the matrices $\tilde{\boldsymbol{\Lambda}}$, $\boldsymbol{U}$, $\boldsymbol{F}$ and $\boldsymbol{Y}$ as follows:

$$\tilde{\boldsymbol{\Lambda}}=\begin{bmatrix}\tilde{\boldsymbol{\Lambda}}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \sigma^2\boldsymbol{I}_{M-K}\end{bmatrix},\quad \boldsymbol{U}=\begin{bmatrix}\boldsymbol{U}_{11} & \boldsymbol{U}_{12}\\ \boldsymbol{U}_{21} & \boldsymbol{U}_{22}\end{bmatrix};\quad \boldsymbol{F}=\begin{bmatrix}\boldsymbol{F}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{F}_2\end{bmatrix},\quad \boldsymbol{Y}=\begin{bmatrix}\boldsymbol{Y}_{11} & \boldsymbol{Y}_{12}\\ \boldsymbol{Y}_{21} & \boldsymbol{Y}_{22}\end{bmatrix}. \qquad (\mathrm{A.23})$$

[3] Let z_1,z_2,z_3,z_4 be jointly circular Gaussian random variables with zero mean. Then (refer to [15]) $E[z_1z_2^*z_3^*z_4]=E[z_1z_2^*]E[z_4z_3^*]+E[z_1z_3^*]E[z_4z_2^*]$.

Here $\tilde{\Lambda}_1, U_{11}, F_1$ and Y_{11} are of sizes $K \times K$, etc. With (A.23) in (A.22) and after some algebraic manipulations and retaining only those terms of order less than or equal to $1/\sqrt{N}$, we have

$$\begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \sigma^2 I_{M-K} \end{bmatrix} + \frac{1}{\sqrt{N}} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \sigma^2 Y_{22} Y_{22}^{\dagger} \end{bmatrix}$$
$$+ \frac{1}{\sqrt{N}} \left\{ \begin{bmatrix} \tilde{\Lambda}_1 W_{11}^{\dagger} & \tilde{\Lambda}_1 W_{21}^{\dagger} \\ \sigma^2 Y_{22} W_{12}^{\dagger} & 0 \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & Y_{22} F_2 Y_{22}^{\dagger} \end{bmatrix} + \begin{bmatrix} W_{11} \tilde{\Lambda}_1 & \sigma^2 W_{12} Y_{22}^{\dagger} \\ W_{21} \tilde{\Lambda}_1 & 0 \end{bmatrix} \right\} + o(1/N), \quad \text{(A.24)}$$

where

$$W_{11} = \sqrt{N}(Y_{11} - I_K), \quad \text{(A.25)}$$

$$W_{12} = \sqrt{N}\, Y_{12}, \qquad W_{21} = \sqrt{N}\, Y_{21}, \quad \text{(A.26)}$$

and the column vectors $w_1, w_2, \ldots, w_K$ are defined to be

$$\begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} = [w_1, w_2, \ldots, w_K] \triangleq W. \quad \text{(A.27)}$$

Similarly,

$$YY^{\dagger} = I_M = \begin{bmatrix} I_K & 0 \\ 0 & Y_{22} Y_{22}^{\dagger} \end{bmatrix} + \frac{1}{\sqrt{N}} \left\{ \begin{bmatrix} W_{11} & W_{12} Y_{22}^{\dagger} \\ W_{21} & 0 \end{bmatrix} + \begin{bmatrix} W_{11}^{\dagger} & W_{21}^{\dagger} \\ Y_{22} W_{12}^{\dagger} & 0 \end{bmatrix} \right\} + o(1/N). \quad \text{(A.28)}$$

Thus, asymptotically for sufficiently large N, from (A.28) and (A.24) we have

$$0 = W_{11} + W_{11}^{\dagger}, \quad \text{(A.29)}$$

$$W_{21} + Y_{22} W_{12}^{\dagger} = 0, \quad \text{(A.30)}$$

$$U_{11} = W_{11} \tilde{\Lambda}_1 + F_1 + \tilde{\Lambda}_1 W_{11}^{\dagger}, \quad \text{(A.31)}$$

and

$$U_{21} = W_{21} \tilde{\Lambda}_1 + \sigma^2 Y_{22} W_{12}^{\dagger} = W_{21} \tilde{\Lambda}_1 - \sigma^2 W_{21}. \quad \text{(A.32)}$$

Since $y_{ii} \geq 0$, this together with (A.25) and (A.29) implies

$$w_{ii} = 0, \quad i = 1, 2, \ldots, K \qquad \text{and} \qquad w_{ij} = -w_{ji}^{*}, \quad i, j = 1, 2, \ldots, K, \quad i \neq j$$

which when substituted into (A.31)-(A.32) gives

$$f_{ii} = u_{ii}, \quad i = 1, 2, \ldots, K, \quad \text{(A.33)}$$

$$w_{ij} = \begin{cases} \dfrac{u_{ij}}{(\tilde{\lambda}_j - \tilde{\lambda}_i)} & i, j = 1, 2, \ldots, K, \quad i \neq j, \\[2ex] \dfrac{u_{ij}}{\tilde{\lambda}_j - \sigma^2} & i = K+1, K+2, \ldots, M, \quad j = 1, 2, \ldots, K. \end{cases} \quad \text{(A.34)}$$

From (A.11) and (A.6) we also have

$$G = \sqrt{N}(C - \tilde{B}) = \sqrt{N}\,\tilde{B}(Y - I) = \sqrt{N}\,\tilde{B} \begin{bmatrix} Y_{11} - I_K & Y_{12} \\ Y_{21} & Y_{22} - I_{M-K} \end{bmatrix},$$

which gives

$$[\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_K] = \tilde{\boldsymbol{B}}[\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_K] \quad \text{or} \quad \boldsymbol{g}_i = \sqrt{N}(\boldsymbol{c}_i - \tilde{\boldsymbol{\beta}}_i) = \tilde{\boldsymbol{B}}\boldsymbol{w}_i, \quad i = 1, 2, \ldots, K.$$

This together with (A.10) and (A.33) gives

$$\tilde{l}_i = \tilde{\lambda}_i + (1/\sqrt{N}) f_{ii} = \tilde{\lambda}_i + (1/\sqrt{N}) u_{ii}, \quad i = 1, 2, \ldots, K, \tag{A.35}$$

and

$$\boldsymbol{c}_i = \tilde{\boldsymbol{\beta}}_i + (1/\sqrt{N})\tilde{\boldsymbol{B}}\boldsymbol{w}_i = \tilde{\boldsymbol{\beta}}_i + (1/\sqrt{N}) \sum_{\substack{j=1 \\ j \neq i}}^{M} w_{ji}\tilde{\boldsymbol{\beta}}_j, \quad i = 1, 2, \ldots, K. \tag{A.36}$$

Thus, the estimators $\tilde{l}_i$ and $\boldsymbol{c}_i$, $i = 1, 2, \ldots, K$, corresponding to the distinct eigenvalues of $\tilde{\boldsymbol{R}}$, are asymptotically multivariate Gaussian random variables/vectors with mean values $\tilde{\lambda}_i$ and $\tilde{\boldsymbol{\beta}}_i$, $i = 1, 2, \ldots, K$, respectively. Furthermore,

$$\operatorname{Cov}(\tilde{l}_i, \tilde{l}_j) = \frac{1}{N} E[u_{ii}u_{jj}] = \frac{1}{N} \tilde{\Gamma}_{ijji}, \quad i, j = 1, 2, \ldots, K \tag{A.37}$$

and

$$\operatorname{Cov}(\boldsymbol{c}_i, \boldsymbol{c}_j) = \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq j}}^{M} E[w_{ki}w_{lj}^*]\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^\dagger = \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq j}}^{M} \frac{\tilde{\Gamma}_{klji}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)} \tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^\dagger. \tag{A.38}$$

Notice that $\boldsymbol{c}_i$ *in* (A.36) *are not normalized vectors*, and it may be emphasized that in the case of eigenvectors, the above asymptotic joint Gaussian property only holds good for these specific sets of unnormalized sample estimators. However, from (A.5) since the eigenvectors $\boldsymbol{c}_i$, $i = 1, 2, \ldots, K$, appearing in (A.8) are normalized ones, to make use of the explicit forms given by (A.36) there, we proceed to normalize these vectors. Starting from (A.34), we have

$$\|\boldsymbol{c}_i\|^2 = \boldsymbol{c}_i^\dagger \boldsymbol{c}_i = 1 + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{M} |w_{ji}|^2 > 1, \tag{A.39}$$

and, hence, the corresponding normalized eigenvectors $\hat{\boldsymbol{c}}_i$, $i = 1, 2, \ldots, K$ have the form

$$\begin{aligned}
\hat{\boldsymbol{c}}_i \triangleq \|\boldsymbol{c}_i\|^{-1}\boldsymbol{c}_i &= \left(1 + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{M} |w_{ji}|^2\right)^{-1/2} \left(\tilde{\boldsymbol{\beta}}_i + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{M} w_{ji}\tilde{\boldsymbol{\beta}}_j\right) \\
&= \left(1 - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^{M} |w_{ji}|^2\right)\tilde{\boldsymbol{\beta}}_i + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{M} w_{ji}\tilde{\boldsymbol{\beta}}_j - \frac{1}{2N\sqrt{N}} \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq i}}^{M} |w_{ki}|^2 w_{li}\tilde{\boldsymbol{\beta}}_l + o(1/N^2).
\end{aligned} \tag{A.40}$$

Using (A.34) and (A.18) we have

$$\begin{aligned}
E[\hat{\boldsymbol{c}}_i] &= \tilde{\boldsymbol{\beta}}_i - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^{M} E[|w_{ji}|^2]\tilde{\boldsymbol{\beta}}_i + o(1/N^2) \\
&= \tilde{\boldsymbol{\beta}}_i - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{\tilde{\Gamma}_{iijj}}{(\tilde{\lambda}_i - \tilde{\lambda}_j)^2} \tilde{\boldsymbol{\beta}}_i + o(1/N^2), \quad i = 1, 2, \ldots, K,
\end{aligned} \tag{A.41}$$

since from the asymptotic joint normal distribution of these zero-mean random variables $u_{ij}(i \neq j)$, their odd-order moments are zero. Thus, asymptotically these normalized estimates for the eigenvectors $\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2, \ldots, \tilde{\boldsymbol{\beta}}_K$, of $\tilde{\boldsymbol{R}}$ are unbiased and the exact bias expressions are given by (A.41). Furthermore, from (A.40)

$$\begin{aligned}\hat{c}_i\hat{c}_j^{\dagger} = {} & \left[1 - \frac{1}{2N}\left(\sum_{\substack{k=1\\k\neq i}}^{M} |w_{ki}|^2 + \sum_{\substack{k=1\\k\neq j}}^{M} |w_{kj}|^2\right)\right]\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\dagger} \\ & + \frac{1}{\sqrt{N}}\left(\sum_{\substack{k=1\\k\neq i}}^{M} w_{ki}\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_j^{\dagger} + \sum_{\substack{k=1\\k\neq j}}^{M} w_{kj}^{*}\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_k^{\dagger}\right) + \frac{1}{N}\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} w_{ki}w_{lj}^{*}\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^{\dagger} \\ & - \frac{1}{2N\sqrt{N}}\left(\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq i}}^{M} |w_{ki}|^2 w_{li}\tilde{\boldsymbol{\beta}}_l\tilde{\boldsymbol{\beta}}_j^{\dagger} + \sum_{\substack{k=1\\k\neq j}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} |w_{kj}|^2 w_{lj}^{*}\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_l^{\dagger}\right. \\ & \left. + \sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} w_{ki}|w_{lj}|^2\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_j^{\dagger} + \sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} |w_{ki}|^2 w_{lj}^{*}\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_l^{\dagger}\right) + o(1/N^2). \qquad \text{(A.42)}\end{aligned}$$

Again, neglecting terms of order $1/N^2$ and proceeding as above, this expression reduces to

$$\begin{aligned}E[\hat{c}_i\hat{c}_j^{\dagger}] & = \tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\dagger} - \frac{1}{2N}\left(\sum_{\substack{k=1\\k\neq i}}^{M} E[|w_{ki}|^2] + \sum_{\substack{k=1\\k\neq j}}^{M} E[|w_{kj}|^2]\right)\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\dagger} + \frac{1}{N}\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} E[w_{ki}w_{lj}^{*}]\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^{\dagger} + o(1/N^2) \\ & = \tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\dagger} - \frac{1}{2N}\left[\sum_{\substack{k=1\\k\neq i}}^{M} \frac{\tilde{\Gamma}_{iikk}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)^2} + \sum_{\substack{k=1\\k\neq j}}^{M} \frac{\tilde{\Gamma}_{jjkk}}{(\tilde{\lambda}_j - \tilde{\lambda}_k)^2}\right]\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\dagger} \\ & \quad + \frac{1}{N}\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} \frac{\tilde{\Gamma}_{klji}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)}\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^{\dagger} + o(1/N^2), \qquad \text{(A.43)}\end{aligned}$$

and proceeding in a similar manner and using (A.21)

$$\begin{aligned}E[\hat{c}_i\hat{c}_j^{\mathrm{T}}] = {} & \tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\mathrm{T}} - \frac{1}{2N}\left[\sum_{\substack{k=1\\k\neq i}}^{M} \frac{\tilde{\Gamma}_{iikk}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)^2} + \sum_{\substack{k=1\\k\neq j}}^{M} \frac{\tilde{\Gamma}_{jjkk}}{(\tilde{\lambda}_j - \tilde{\lambda}_k)^2}\right]\tilde{\boldsymbol{\beta}}_i\tilde{\boldsymbol{\beta}}_j^{\mathrm{T}} \\ & + \frac{1}{N}\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq j}}^{M} \frac{\tilde{\Gamma}_{kjli}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)}\tilde{\boldsymbol{\beta}}_k\tilde{\boldsymbol{\beta}}_l^{\mathrm{T}} + o(1/N^2). \qquad \text{(A.44)}\end{aligned}$$

An easy verification shows that $\mathrm{Cov}(\hat{c}_i, \hat{c}_j)$ is once again given by (A.38), but nevertheless, (A.42)-(A.43) will turn out to be useful in computing the asymptotic bias and variance of the sample direction estimator $\hat{Q}(\omega)$ in (13).

The conventional MUSIC scheme [16] now follows as a special case of this analysis. In that case, from (A.16)-(A.18) retaining only the terms corresponding to the forward array, we have

$$\Gamma_{iklj} = \boldsymbol{\beta}_i^{\dagger}\boldsymbol{R}\boldsymbol{\beta}_k\boldsymbol{\beta}_l^{\dagger}\boldsymbol{R}\boldsymbol{\beta}_j = \lambda_i\lambda_j\delta_{ik}\delta_{jl}. \qquad \text{(A.45)}$$

Thus, for the conventional MUSIC case using (A.45), we obtain

$$E[\hat{\boldsymbol{c}}_i] = \boldsymbol{\beta}_i - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \boldsymbol{\beta}_i + o(1/N^2), \tag{A.46}$$

$$E[\hat{\boldsymbol{c}}_i \hat{\boldsymbol{c}}_j^\dagger] = \boldsymbol{\beta}_i \boldsymbol{\beta}_j^\dagger - \frac{1}{2N} \left[\sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} + \sum_{\substack{k=1 \\ k \neq j}}^{M} \frac{\lambda_j \lambda_k}{(\lambda_j - \lambda_k)^2} \right] \boldsymbol{\beta}_i \boldsymbol{\beta}_j^\dagger$$

$$+ \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} \boldsymbol{\beta}_k \boldsymbol{\beta}_k^\dagger \delta_{ij} + o(1/N^2), \tag{A.47}$$

and

$$E[\hat{\boldsymbol{c}}_i \hat{\boldsymbol{c}}_j^{\mathrm{T}}] = \boldsymbol{\beta}_i \boldsymbol{\beta}_j^{\mathrm{T}} - \frac{1}{2N} \left[\sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} + \sum_{\substack{k=1 \\ k \neq j}}^{M} \frac{\lambda_j \lambda_k}{(\lambda_j - \lambda_k)^2} \right] \boldsymbol{\beta}_i \boldsymbol{\beta}_j^{\mathrm{T}} - \frac{1}{N} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \boldsymbol{\beta}_j \boldsymbol{\beta}_i^{\mathrm{T}} (1 - \delta_{ij}) + o(1/N^2),$$

where $\lambda_i, \boldsymbol{\beta}_i$, $i = 1, 2, \ldots, M$ are as defined in (7).

Once again for the f/b case, using (A.8)-(A.9) and recalling that the eigenvector estimators appearing there are normalized ones, we have

$$y_i(\omega) = |\hat{\boldsymbol{c}}_i^\dagger \boldsymbol{a}(\omega)|^2 = \boldsymbol{a}^\dagger(\omega) \hat{\boldsymbol{c}}_i \hat{\boldsymbol{c}}_i^\dagger \boldsymbol{a}(\omega) \tag{A.48}$$

and from (A.43) and (13) we have

$$E[\hat{Q}(\omega)] = \tilde{Q}(\omega) + \frac{1}{N} \sum_{i=1}^{K} \left[\sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{\tilde{\Gamma}_{iikk}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)^2} |\tilde{\boldsymbol{\beta}}_i^\dagger \boldsymbol{a}(\omega)|^2 \right.$$

$$\left. - \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq i}}^{M} \frac{\tilde{\Gamma}_{iikl}}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_i - \tilde{\lambda}_l)} \boldsymbol{a}^\dagger(\omega) \tilde{\boldsymbol{\beta}}_k \tilde{\boldsymbol{\beta}}_l^\dagger \boldsymbol{a}(\omega) \right] + o(1/N^2), \tag{A.49}$$

and this shows that

$$E[\hat{Q}(\omega)] \to \tilde{Q}(\omega) \text{ as } N \to \infty.$$

Also,

$$\mathrm{Var}(\hat{Q}(\omega)) = E[\hat{Q}^2(\omega)] - (E[\hat{Q}(\omega)])^2 = E\left[\left(1 - \sum_{i=1}^{K} y_i \right)^2 \right] - \left(E\left[\left(1 - \sum_{i=1}^{K} y_i \right) \right] \right)^2$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} (E[y_i y_j] - E[y_i] E[y_j]). \tag{A.50}$$

Using (A.48) and (A.42), after a series of manipulations, we have

$$E[y_i y_j] - E[y_i] E[y_j] = \frac{1}{N} E[d_i(\omega) d_j(\omega)] + o(1/N^2),$$

where

$$d_i(\omega) = \sum_{\substack{k=1 \\ k \neq i}}^{M} \left(w_{ki} \boldsymbol{a}^\dagger(\omega) \tilde{\boldsymbol{\beta}}_k \tilde{\boldsymbol{\beta}}_i^\dagger \boldsymbol{a}(\omega) + w_{ki}^* \boldsymbol{a}^\dagger(\omega) \tilde{\boldsymbol{\beta}}_i \tilde{\boldsymbol{\beta}}_k^\dagger \boldsymbol{a}(\omega) \right),$$

which gives

$$E[y_i y_j] - E[y_i]E[y_j] = \frac{2}{N} \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq j}}^{M} \frac{\mathrm{Re}[\{\tilde{\Gamma}_{klji} \boldsymbol{a}^{\dagger}(\omega) \tilde{\boldsymbol{\beta}}_j \tilde{\boldsymbol{\beta}}_l^{\dagger} \boldsymbol{a}(\omega) + \tilde{\Gamma}_{kjli} \boldsymbol{a}^{\dagger}(\omega) \tilde{\boldsymbol{\beta}}_l \tilde{\boldsymbol{\beta}}_j^{\dagger} \boldsymbol{a}(\omega)\} \boldsymbol{a}^{\dagger}(\omega) \tilde{\boldsymbol{\beta}}_k \tilde{\boldsymbol{\beta}}_i^{\dagger} \boldsymbol{a}^{\dagger}(\omega)]}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)}$$

$$+ \mathrm{o}(1/N^2). \tag{A.51}$$

Finally, with (A.51) in (A.50) we get

$$\mathrm{Var}(\hat{Q}(\omega)) = \frac{2}{N} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{\substack{k=1 \\ k \neq i}}^{M} \sum_{\substack{l=1 \\ l \neq j}}^{M} \frac{\mathrm{Re}[\{\tilde{\Gamma}_{klji} \boldsymbol{a}^{\dagger}(\omega) \tilde{\boldsymbol{\beta}}_j \tilde{\boldsymbol{\beta}}_l^{\dagger} \boldsymbol{a}(\omega) + \tilde{\Gamma}_{kjli} \boldsymbol{a}^{\dagger}(\omega) \tilde{\boldsymbol{\beta}}_l \tilde{\boldsymbol{\beta}}_j^{\dagger} \boldsymbol{a}(\omega)\} \boldsymbol{a}^{\dagger}(\omega) \tilde{\boldsymbol{\beta}}_k \tilde{\boldsymbol{\beta}}_i^{\dagger} \boldsymbol{a}^{\dagger}(\omega)]}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)}$$

$$+ \mathrm{o}(1/N^2) \to 0 \quad \text{as } N \to \infty. \tag{A.52}$$

Thus $\hat{Q}(\omega)$ is a consistent estimator in all cases.

Appendix B. Equivalence of eigenvectors

Let $\tilde{\boldsymbol{R}}$ be an $M \times M$ Hermitian matrix with distinct eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_r$, where $m_1, m_2, \ldots, m_r$, represent their repetitions. Then $m_1 + m_2 + \cdots + m_r = M$; further, let $\tilde{\boldsymbol{\beta}}_{11}, \tilde{\boldsymbol{\beta}}_{12}, \ldots, \tilde{\boldsymbol{\beta}}_{1m_1}, \ldots, \tilde{\boldsymbol{\beta}}_{1M}$, represent one set of associated normalized eigenvectors. With

$$\tilde{\boldsymbol{B}}_1 = [\tilde{\boldsymbol{\beta}}_{11}, \tilde{\boldsymbol{\beta}}_{12}, \ldots, \tilde{\boldsymbol{\beta}}_{1M}], \qquad \tilde{\boldsymbol{B}}_1 \tilde{\boldsymbol{B}}_1^{\dagger} = \boldsymbol{I}_M$$

and

$$\tilde{\boldsymbol{\Lambda}} = \mathrm{diag}[\tilde{\lambda}_1, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_r, \ldots, \tilde{\lambda}_r],$$

we now have

$$\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{B}}_1 \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{B}}_1^{\dagger}. \tag{B.1}$$

Let

$$\tilde{\boldsymbol{B}}_2 = [\tilde{\boldsymbol{\beta}}_{21}, \tilde{\boldsymbol{\beta}}_{22}, \ldots, \tilde{\boldsymbol{\beta}}_{2M}]$$

represent yet another set of normalized eigenvectors of $\tilde{\boldsymbol{R}}$. Then

$$\tilde{\boldsymbol{B}}_2 \tilde{\boldsymbol{B}}_2^{\dagger} = \boldsymbol{I}_M, \qquad \tilde{\boldsymbol{R}} = \tilde{\boldsymbol{B}}_2 \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{B}}_2^{\dagger}, \tag{B.2}$$

and from (B.1) and (B.2) we have

$$\tilde{\boldsymbol{B}}_1 \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{B}}_1^{\dagger} = \tilde{\boldsymbol{B}}_2 \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{B}}_2^{\dagger},$$

or equivalently

$$\tilde{\boldsymbol{\Lambda}} \boldsymbol{V} = \boldsymbol{V} \tilde{\boldsymbol{\Lambda}}, \tag{B.3}$$

where

$$\boldsymbol{V} = \tilde{\boldsymbol{B}}_1^{\dagger} \tilde{\boldsymbol{B}}_2. \tag{B.4}$$

Thus, $\boldsymbol{V}$ is also unitary and, further, $\tilde{\boldsymbol{\Lambda}}$ and $\boldsymbol{V}$ commute. Moreover, from (B.3) we have

$$\tilde{\lambda}_i v_{ij} = v_{ij} \tilde{\lambda}_j,$$

which for $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ gives $v_{ij} = 0$. This together with the fact that V is unitary implies V is block unitary with blocks of sizes $m_1, m_2, \ldots, m_r$: hence from (B.4) we have

$$\boldsymbol{B}_2 = \tilde{\boldsymbol{B}}_1 \boldsymbol{V} = \tilde{\boldsymbol{B}} \begin{bmatrix} \boldsymbol{V}_1 & & & \boldsymbol{0} \\ & \boldsymbol{V}_2 & & \\ & & \ddots & \\ \boldsymbol{0} & & & \boldsymbol{V}_r \end{bmatrix}, \tag{B.5}$$

where

$$\boldsymbol{V}_i \boldsymbol{V}_i^{\dagger} = \boldsymbol{I}_{m_i}, \quad i = 1, 2, \ldots, r.$$

Notice that in the special case, when all eigenvalues of $\tilde{\boldsymbol{R}}$ are distinct, then $\boldsymbol{V}$ is diagonal and unitary and each diagonal entry is a phase factor. In that case

$$\tilde{\boldsymbol{\beta}}_{1i} = e^{j\phi_i} \tilde{\boldsymbol{\beta}}_{2i}, \quad i = 1, 2, \ldots, M. \tag{B.6}$$

In particular, different sets of signal subspace normalized eigenvectors of $\tilde{\boldsymbol{S}}$ are related in this fashion (see (A.7)).

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